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THE MATHEMATICS STUDENT

Volume V] JUNE 1937 [Number

QUATERNIONS AND QUANTUM MECHANICS

BY

M. RAZIUDDIN SIDDIQI, *Osmania University, Hyderabad (Deccan)*

It is a truism that Mathematics is becoming more and more indispensable in the Natural Sciences, and specially in Physics. As our knowledge of the world advances, even the most abstract branches of Mathematics are finding applications. We have gradually seen the functions of complex variables, the differential and integral equations, tensors, matrices and groups become powerful tools in the hands of physicists. Quaternions, which were at one time considered incapable of finding any application, had already proved of value in some branches of physics. But recently a discovery has been made which bids fair to bring them to the forefront.

It is well-known that although the non-relativistic theory of quantum mechanics has attained a certain completeness, grave difficulties are encountered when it is tried to reconcile it to the theory of relativity. Incessant attempts to build up a relativistic quantum mechanics have been made during the last ten years, the most successful being that of Dirac. Dirac's work, however, is limited to a single electron, and its extension to general systems has not yet been possible.

M. Bernard Kwal, who is working in the Institut Henri Poincaré, under Prof. Louis de Broglie, has lately published his researches in the *Comptes Rendus* and *Journal de Physique* which show that there is no incompatibility between the principles of quantum mechanics and those of relativity, and that the results of quantum mechanics can be easily generalised into a relativistic theory provided we employ quaternions.

M. Kwal remarks that from the point of view of the mathematical description of phenomena, the world point of space time should be regarded as a parameter, and the physical quantities as functions of

this parameter. In the existing theories only time is considered as the parameter, whereas according to the relativity theory the three co-ordinates of space and the co-ordinate of time all have the same status. The physical quantities are thus functions of the world-point (x, y, z, ict) which is taken as the quaternionian variable with the four components:

$$X = (x, y, z, ict).$$

The use of quaternions has the advantage of preserving the four-dimensional character of physical quantities all along the calculations.

Accordingly the dynamical variables are not the Heisenberg matrices

$$q = \left(q_{mn}\, e^{-2\pi i \nu_{mn} t}\right),$$

but the quaternionian matrices

$$\bar{q} = \left(\bar{q}_{mn}\, e^{2\pi i \left(\bar{X} \nu_{mn}/\bar{c}\right)}\right) = \left(\bar{q}_{mn} e^{2\pi i \left(x \frac{\nu_{mn}}{c} + y \frac{\nu_{mn}}{c} + z \frac{\nu_{mn}}{c} - \nu_{mn} t\right)}\right).$$

where $\bar{X}$ and $\nu_{mn}/\bar{c}$ are the two quaternions

$$\bar{X} \quad (x, y, z, ict),$$

$$\frac{\nu_{mn}}{\bar{c}} \quad \left(\frac{\nu_{mn}}{c}\alpha, \frac{\nu_{mn}}{c}\beta, \frac{\nu_{mn}}{c}\gamma, i\frac{\nu_{mn}}{c}\right).$$

The q-numbers and the c-numbers of the quantum mechanics thus become quaternions.

The quaternionian algebra and analysis developed by Hamilton and Tait furnish the necessary apparatus for generalising the quantum conditions, the equations of motion and the uncertainty relations of Heisenberg, and the Ritz combination Principle.

He shows further that the principle of relativity in the quaternionian form and the quantum postulate lead very simply to Dirac's wave-equation. Suppose that a dynamical system is characterised on the macroscopic scale by the four-dimensional energy-momentum vector P (quaternion), which is a function of the quaternionian parameter x. Then the quantum postulate is that the passage to the microscopic scale is effected by replacing P by the differential operator $\frac{p}{2\pi i}\frac{\partial}{\partial X} + \frac{e}{c}A$, where A is the electromagnetic potential.

Now, the simplest relativistic property of P is that in the proper system of co-ordinates it has the value imc. Then the most general

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Lorentz-transformation which allows us to pass from the co-ordinate-system of the observer to the proper system is given by

$$\overset{*}{S}^{-1}\, P\, S = i\, m\, c,$$

where $\overset{*}{S}$ and S are the semi-vectors of Einstein and Meyer's unified field theory which satisfy the relations

$$i\, \overset{*}{S}\, S^{-1} = u/c, \quad u = v \Big/ \sqrt{1 - \frac{v^2}{c^2}},$$

v being the velocity of the particle relative to the observer. This transformation equation is just the expression of Dirac's wave-equation in the quaternionian form.

It seems that quaternions are destined to play an increasingly important role in modern physics.

A THEOREM ON BINARY QUARTICS AND ITS APPLICATION TO GEOMETRY

BY

B. RAMAMURTI, *Annamalai University.*

The object of this note is to prove a theorem on binary quartics and point out its consequences in the familiar geometric representations of binary quartics.

The pencil of quartics determined by a quartic $a_x^4 \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4$, and its Hessian b_x^4 is called a syzegetic pencil. The three mutually apolar involutions which carry a_x^4 into itself also carry b_x^4 into itself, and hence every member of the syzegetic pencil into itself. The double points of the three involutions form three mutually apolar quadratics p_x^2, q_x^2, r_x^2. The sextic $B_x^6 \equiv p_x^2\, q_x^2\, r_x^2$ is the Jacobian sextic of the pencil. Following Dr. Vaidyanathaswamy, let us call such a sextic a remarkable sextic. We shall now prove

THEOREM 1. *If c_x^4 be apolar * to every member of a syzegetic pencil $a_x^4 + \lambda b_x^4$, the Jacobian of c_x^4 with any d_x^4, is apolar to the Jacobian of the syzegetic pencil. Conversely if a sextic α_x^6 is apolar to a remarkable sextic β_x^6, there must be at least one member in any pencil of quartics having α_x^6 as the † Jacobian, apolar to every member of the syzegetic pencil determined by β_x^6.*

Direct Proof.

Any quartic a_x^4 can be reduced by a linear transformation to the canonical form $x^4 + 6kx^2 + 1$. The three mutually apolar involutions carrying it into itself are

$I_1(x) = 1/x \quad I_2(x) = -x \quad I_3(x) = -1/x.$ The syzegetic pencil is obtained by varying λ in

$$x^4 + 6\lambda x^2 + 1. \qquad \ldots \quad (1)$$

The Jacobian of the pencil is $x(x^4 - 1)$. ... (2)

---

* Two binary forms of order n

$$a_x^n = a_0\, x^n + {}_nc_1\, a_1\, x^{n-1} \quad + {}_nc_{n-1}\, a_{n-1}\, x + a_n$$
$$b_x^n = b_0\, x^n + {}_nc_1\, b_1\, x^{n-1} \quad + {}_nc_{n-1}\, b_{n-1}\, x + b_n$$

are said to be apolar if $a_0\, b_n - nc_1\, a_1\, b_{n-1} + \cdots (-1)^n\, a_n\, b_0 = 0.$

† An extensive study of the syzegetic pencil of quartics and various other special pencils of quartics has been made by Dr. R. Vaidyanathaswamy

(1) Linear systems of binary cubics and quadrics. Pamphlet published for private circulation 1921.

(2) The null pencil of binary quartics. *Proc. London Math. Soc. 23, 317-325.*

(3) A special pencil of binary quartics. *Proc. Edinburgh Math. Soc. 1 (new series) 104-110.*

Let $\quad c_x^4 \equiv c_0 x^4 + 4c_1 x^3 + 6c_2 x^2 + 4c_3 x + c_4$

$\quad d_n^4 \equiv d_0 x^4 + 4d_1 x^3 + 4d_2 x^2 + 4d_3 x + d_4.$

Let their Jocobian * be

$$\alpha_x^6 \equiv \alpha_0 x^6 + 6c_1 \alpha_1 x^5 + 6c_2 \alpha_2 x^4 + 6c_3 \alpha_3 x^3 + 6c_2 \alpha_4 x^2 + 6c_1 \alpha_5 x + \alpha_6 = 0.$$

The apolar invariant of (2) and (3) is $\alpha_1 - \alpha_5$. ... (3)

Calculating the Jacobian of c_x^4 and d_x^4 we find

$$\alpha_1 = \tfrac{1}{2}(c_2 d_0 - c_0 d_2)$$
$$\alpha_5 = -\tfrac{1}{2}(c_2 d_4 - c_4 d_2)$$

Hence the apolar invariant is $\frac{1}{2}\{c_2(d_0 + d_4) - d_2(c_0 + c_4)\}$ ... (4)

Suppose c_x^4 is apolar to every member of (1). This requires that $c_2 = 0$ and $c_0 + c_4 = 0$. Hence (4) vanishes. The two sextics are apolar.

Conversely let the two sextics be apolar so that

$$c_2(d_0 + d_4) - d_2(c_0 + c_4) = 0 \qquad \dots \quad (5)$$

Let us take the member $f_x^4 = c_2 d_x^4 - d_2 c_x^4$. Then $f_2 = 0$ and $f_1 + f_4 = 0$ from (5). Hence f_x^4 is apolar to every member of (1).

Symbolic proof.

$$\text{Given } (ca^4) = (cb)^4 = 0. \qquad \dots \quad (6)$$

Since b_x^4 is the Hessian of a_x^4 therefore $(ab)^3 a_x b_x \equiv 0$. ... (7)

The apolar invariant of the Jacobian of the two pencils is

$$\{(ab)^3 a_x^3 b_x^3, \quad (cd) c_x^3 d_x^3)\}^6$$
$$= \mathrm{R}(\mathrm{P}^3 - \mathrm{Q}^3 + 9\ \overline{\mathrm{PQ}^2 - \mathrm{P}^2\mathrm{Q}}) \qquad \dots \quad (8)$$

where $\quad \mathrm{R} = (ab)(cd), \quad \mathrm{P} = (bc)(ad), \quad \mathrm{Q} = (ca)(bd)$

By repeated use of the identity $\mathrm{P} + \mathrm{Q} + \mathrm{R} \equiv 0$ we have

$$5(\mathrm{P}^4 - \mathrm{Q}^4) + 4\mathrm{R}^3(\mathrm{P} - \mathrm{Q})$$
$$= 5(bc)^4 (ad)^4 - 5(ca)^4 (bd)^4$$
$$+ 4(ab)^3 (cd)^3 [(bc)(ad) + (ac)(bd)]$$
$$= 0 \quad \text{from (6) and (7).} \qquad \dots \quad (9)$$

Conversely let $\quad \alpha_x^6 = (cd) c_x^3 d_x^3 \quad \beta_x^6 = (ab) a_x^3 b_x^3$ and $(\alpha\beta)^6 = 0$.

Choose c_x^4 so that it is apolar to a_x^4, then $(ca)^4 = 0$. Hence from (7) and (9) either $(bc)^4 = 0$ or $(ad)^4 = 0$. In the first case c_x^4 is apolar to every member of the first pencil. In the second case d_x^4 is apolar

* The Jacobian of two forms a_x^n, b_x^n is the form of order $(2n-2)$ whose roots are the repeated roots of members of the pencil $a_x^n + \lambda b_x^n$. It is given by

$$\begin{vmatrix} a_x^n & b_x^n \\ \dfrac{d}{d_x}(a_x^n) & \dfrac{d}{d_x}(b_x^n) \end{vmatrix}$$

to a_x^4. Hence every member of the first pencil is apolar to d_x^4, and then we can find a member, which in addition is also apolar to b_x^4. Then it will be apolar to every member of the second pencil. Thus the converse is true.

2. Geometric interpretation in the plane.

If a conic C be regarded as the carrier of a binary variable x, any quadratic p_x^2 can be represented uniquely by the point of intersection of the tangents of C at the points given parametrically by $p_x^2=0$. Any binary quartic c_x^4 can be represented by the unique conic $Q(c_x^4)$ out-polar to C and cutting it at the points given parametrically by $c_x^4=0$. A remarkable sextic β_x^6, consisting of three apolar quadratics $p_x^2\, q_x^2\, r_x^2$ can be represented by a self-conjugate triangle PQR whose vertices correspond to the three quadratics. Any quartic apolar to the syzegetic pencil determined by β_x^6 corresponds to a conic circumscribed to PQR and hence out-polar to C *. Hence if α_x^6 is apolar to β_x^6 and we take a pencil of out-polar conics corresponding to a pencil of quartics having α_x^6 as the Jacobian sextic, from Theorem 1, there must be at least one member circumscribed to PQR. Conversely if we take any conic of the pencil, there are ∞^1 triangles inscribed in it and self-polar with respect to c. Hence they must correspond to remarkable sextics apolar to α_x^6. Thus we have.

THEOREM 2. *If we take a pencil of out-polar conics having* α_x^6 *as the Jacobian sextic, the* ∞^2 *remarkable sextic* β_x^6 *apolar to* α_x^6 *can be represented by the* ∞^2 *triangles inscribed in the conics of the pencil and self-polar with respect to the base conic.*

But given α_x^6 there are five pencils of quartics having α_x^6 as the Jacobian. Let the out-polar conics corresponding to one of the pencils intersect in four points forming quadrangle, which is self-polar with respect to the conic. This quadrangle, and its polar quadrilateral with respect to C form a D.C. (Desargues configuration) self-polar with respect to C. From the 10 vertices of the D.C. can be formed 5 self-polar quadrangles. According to Stephanos, the pencils of conics $P_1, P_2 \ldots\ldots P_5$ determined by these quadrangles cut out on C, the five pencils of quartics having α_x^6 as the Jacobian sextic. Hence if the remarkable sextic β_x^6 is apolar to α_x^6, in each one of the above five pencils of conics, there must be one conic circumscribed to the self-conjugate triangle PQR determined by βx^6. In other words any self-conjugate triangle PQR inscribed in a conic of one of the pencils, must

* The apolar net is $\lambda\, p_x^2\, q_x^2 + \mu\, q_x^2\, r_x^2 + \nu \cdot r_x^2\, p_x^2$.

also be inscribed in one conic of each of the other four pencils. Hence the conic of P_2 which passes through P must also pass through Q and R. Since conics, one from each of two pencils have also a point of the D.C. in common, we have,

THEOREM 3. *The residual intersections of two out-polar conics, one from each of two Co-Jacobian systems, other than a point of the D.C. form a triangle self-polar with respect to the base conic. Such triangles correspond to the remarkable sextics apolar to the sextic determined by and determining the D.C.* *

3. Geometric interpretation in [3]

If we take a twisted cubic C^3 in [3], any binary cubic a_x^3 can be represented by the unique point of intersection of the osculating planes of C^3 at the points given parametrically by $a_x^3=0$. Any binary sextic γ_x^6 can be represented by the unique quadric locus $Q(\gamma_x^6)$ passing through the points on C^3 given parametrically by $\gamma_x^6=0$ and out-polar to C^3 or by the quadric envelope $\bar{Q}(\gamma_x^6)$ in-polar to C and touching the osculating planes at these points. To a pencil of quartics, can be made to correspond, the unique in polar quadrics for which the pencil of inscribed tetrahedra given parametrically by the pencil of quartics, are self-polar. The quadric, as a locus, intersects C^3 at the points given parametrically by the Jacobian sextic α_x^6 of the pencil. Also it is known if a sextic αx^6 is apolar to β_x^6 the in-polar quadric envelope $\bar{Q}(\beta_x^6)$ is apolar to every quadric through the points α_x^6, and conversely. †

Let us take a remarkable sextic β_x^6 and take the corresponding in-polar quadric envelope. It is known that it is also out-polar and is

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* It is well-known that the ten lines joining any two points and the ten planes through any three of five points in general position in [3] give as section with a plane a D.C. of 10 points and 10 lines. The ∞^4 quadrics through the points give rise to ∞^4 conics in the plane, all apolar to a conic C with respect to which the D.C. is self-polar. The five pencils of cones each containing the four lines through a point give the five pencils of out-polar conics having the same Jacobian sextic α_x^6. Two cones one from each pencil give as section two conics one from each co-Jacobian pencil. The cones have a generator in common and hence intersect besides, in a twisted cubic, which cuts the plane in the points of the self-polar triad PQR. Hence, it follows, that the intersections of the ∞^2 twisted cubics with the plane correspond to the ∞^2 remarkable sextics apolar to α_x^6.

See (1) Conner. *American Journal of Math.* 32. (1910).

(2) Question 1650 *Math. Student* 4. 160. Proposed by V. Ramaswamy Ayer and solved by K. Rangaswamy.

(3) For the D.C.'s admitting a collineation group and the nature of the Co-Jacobian pencils in such cases see Ramamurti. *Journal London Math. Soc. 8.* 34-39.

† Meyer *Apolarität und Rationale Kurven.* Page 200.

determined as a locus by the same sextic. Such a quadric is called a remarkable quadric and has been studied in detail by Dr. Vaidyanathaswamy.* Let us take a tetrahedron inscribed in C^3 and given parametrically by a quartic C_x^4 apolar to the syzegetic pencil determined by βx^6. The ∞^3 pencils of quartics having C_x^4 as a member, can be represented by the ∞^3 quadrics for which the tetrahedron is self-polar. Since from Theorem 1, the Jacobians of such pencils are apolar to βx^6, the quadrics for which the tetrahedron T (cx^4) is self-polar must be out-polar to the remarkable quadric Q (β_x^6). In particular the squared planes forming the faces of T must be outpolar to Q and hence touch it. Hence T must be circumscribed to Q. Thus we have,

THEOREM 4. *There is a net of tetrahedra inscribed in a twisted cubic and circumscribed to a remarkable quadric. If the quadric corresponds to the remarkable sextic* β^6, *the net of tetrahedra are given parametrically by the net of quartics apolar to the syzegetic pencil determined by* β_x^6.

Let α_x^6 be apolar to β_x^6, and let us take a pencil of quartics having α_x^6 as the Jacobian, and represent it by the corresponding in-polar quadric. From Theorem 1 there must be one member C_x^4 apolar to every member of the syzegetic pencil determined by βx^6. Hence there must be one tetrahedron inscribed in C^3, circumscribed to the remarkable quadric and self-polar with respect to the in-polar quadric determined by the pencil. But given α_x^6 there are five pencils of quartics having it as the Jacobian. In other words there are 5 quadrics through the points α_x^6 on C^3, in-polar to C^3. If α_x^6 is apolar to β_x^6, the above relationship must be true for all the quadrics. Let us call these, associated quadrics. Then we have

Taking a system of five associated in-polar quadrics determined by a sextic αx^6, *and a remarkable quadric* Q (βx^9), *if the property of having a self-polar tetrahedron inscribed in the twisted cubic and circumscribed to the remarkable quadric, is true for one of the five quadrics, it is also true for the other four. The necessary and sufficient condition for this is the vanishing of the apolar invariant of the two sextics* αx^6 *and* βx^6.

4. Geometric interpretation in [4]

If we take a rational norm curve C^4 in [4] as the carrier of a binary variable x, any quartic C_x^4 can be represented by the unique point of intersection of the osculating primes of C^4 at the points given

* (4) Vaidyanathaswamy...*Proc. Camb. Phil. Soc.* 26 (1930) 206-219.

parametrically by $C_x^4=0$. Any sextic β_x^6 can be represented by the unique linear complex $L(\beta_x^6)$, containing the tangents of C^4 at the points $\beta_x^6=0$ and outpolar to C^4. The line joining the points C_x^4 and d_x^4 will belong to $L(\beta_x^6)$ if their Jacobian is apolar to β_x^6. Hence if the Jacobian of C_x^4 with every dx^4, is apolar to β_x^6, every line through Cx^4 will belong to the linear complex and hence Cx^4 is a singular point. Hence if βx^6 is a remarkable sextic the point corresponding to every quartic of the net apolar to the syzegetic pencil determined by β_x^6 is a singular point. Hence we have

THEOREM 5. *The outpolar linear complex determined by a remarkable sextic β_x^6 has a plane of singular points which are given parametrically by the net of quartics apolar to the syzegetic pencil determined by β_x^6.**

Also if α_x^6 is apolar to β_x^6, the five pencils of quartics having αx^6 as the Jacobian, must each have a member belonging to the apolar net defined by β_x^0. Hence the corresponding five lines representing the five pencils, must intersect the singular plane of the outpolar linear complex determined by β_x^6. Let us call the five lines, associated lines. We then have

If the plane of singular points of the outpolar linear complex determined by a remarkable sextic intersects any line, it intersects also the other four associated lines.†

In conclusion it may be remarked that most of the results obtained in this paper are well-known. The interesting point is however, that all these are related to one another and depend upon the same fundamental result in binary forms.

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*Taking a pencil of binary quartics, we find that it can be represented by a line in [4] and by an inpolar quadric in [3]. It follows from the results above obtained that the inpolar quadrics outpolar to a remarkable quadric correspond to the lines of an outpolar complex containing a plane of singular points. This is only a special case of the general correspondence between quadrics Q in [n] and linear line complexes L in [$n+1$] discovered by Dr. R. Vaidyanathaswamy in which the inpolar quadrics outpolar to the quadric envelope Q correspond to the lines of the linear complex L.

See. R. Vaidyanathaswamy. On the rational norm curve II *Journal London Math. Soc.* 7 (1932) 52-56.

H. G. Telling. *Three related quartic curves in four dimensions. Proc. Camb. Phil. Soc.* 28 (1932). 401-415. 83-1, 3-2. The latter author had also studied the above correspondence in the special cases $n=2$ and 3, and was probably not aware of the fact that the correspondence had already been discovered and published by the former.

† Compare. H. G. Telling. *Rational Quartic curve.* Cambridge Tracts No. 34, Page 30.

ON SYMMETRY IN THREE DIMENSIONS

BY

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[*Communicated by Dr. Francis Regan, St. Louis University.*]

I. Introduction

An analytic treatment of symmetry with respect to a single center or two or three axes or planes is familiar to any reader of Co-ordinate Geometry of Space. We may, however, have symmetry with respect to more than one center and more than three axes or planes. The following discussion treats these cases from the standpoint of Analysis Situs. A defined set of points having certain elements of symmetry is given. A point of the set is chosen and in terms of this point a typical subset of the given set is determined. The points of the subset are then paired to determine what additional elements of symmetry the given set has. Both in the derivation of the points of the subsets and in the examination of these subsets for elements of symmetry, the ordinary methods of Cartesian Geometry of space were used. The general method is due to Lyle* who studied all the possible cases in two dimensions.

II. Definitions

If the points of a set M may be taken in pairs such that a point *P* is a midpoint between the points of each pair, then P is a *center of symmetry of the set* M.

If the points of a set M may be taken in pairs such that a line L is the perpendicular bisector of each of the segments of lines joining points of a pair, then L is an *axis of symmetry of the set M.*

If the points of a set M may be taken in pairs such that each of the lines joining points of a pair is perpendicular to a plane T, and the segments between the points of a pair are bisected by the plane T, then T is a *plane of Symmetry of the set* M.

III. Theorems concerning sets with perpendicular or coincident elements of symmetry

We first consider sets having two perpendicular or coincident elements of symmetry, and take for simplicity the co-ordinate elements of Cartesian Geometry.

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* Lyle, "Centers and Axes of Symmetry," *Bulletin of the American Mathematical Society*, July, 1933, Abstract No. 207, p. 504.

THEOREM 1.—*If a set M has a center of symmetry, a necessary and sufficient condition that the set shall have an axis of symmetry through the center is that it shall have a plane of symmetry through the center.*

Necessary condition. Let M have the origin as a center of symmetry and the X-axis as an axis of symmetry. If (a, b, c) is a point of M, then since the set must contain a point symmetric to (a, b, c) with respect to each of the given elements of symmetry, M contains $(-a, -b, -c)$ and $(a, -b, -c)$. But if these points belong to M, the set also contains $(-a, b, c)$ and (a, b, c). Taking these in pairs, $(-a, b, c)$ and (a, b, c); and $(-a, -b, -c)$ and $(a, -b, -c)$, we see that these pairs are symmetric with respect to the YZ-plane. Since (a, b, c) was any point of M, any property of the subset of four points derived from it would be a property of all other subsets and hence of the set M.

Sufficient condition. Let M have the origin as a center of symmetry and the YZ-plane as a plane of symmetry. Then if (a, b, c) is a point of M, the points $(-a, -b, -c)$ and $(-a, b, c)$ are points of M. But if these belong to the set, $(a, -b, -c)$ also belongs to the set M. Pairing these thus, (a, b, c) and $(a, -b, -c)$, and $(-a, b, c)$ and $(-a, -b, -c)$ we see that the subset is symmetric with respect to the X-axis. Hence M has the X-axis as an axis of symmetry.

THEOREM 2.—*If a set M has a plane of symmetry, a necessary and sufficient condition that a set shall have a center of symmetry is that the set shall have an axis of symmetry perpendicular to the given plane.*

THEOREM 3.—*If a set M has a plane of symmetry, a necessary and sufficient condition that the set shall have an axis of symmetry in the given plane is that that set shall have another plane of symmetry perpendicular to the given plane.*

We now take up sets having three elements of symmetry which are restricted as above. The theorems may be divided into two groups, the first of which give sufficient conditions that a set be symmetric with respect to all the co-ordinate elements. They are:

(1) the set shall have a center of symmetry and two perpendicular axes of symmetry which pass through the center;

(2) the set shall have a plane of symmetry, an axis of symmetry in the given plane and a center of symmetry on the given axis;

(3) the set shall have two perpendicular planes of symmetry and a center of symmetry on the intersection of the two given planes;

(4) the set shall have an axis of symmetry and two perpendicular planes of symmetry, the axis being contained in one plane and perpendicular to the other;

(5) the set shall have two axes of symmetry and a plane of symmetry perpendicular to one of the axes and containing the other;

(6) the set shall have two axes of symmetry and a plane of symmetry containing the two axes;

(7) the set shall have three mutually perpendicular planes of symmetry.

The second group of theorems gives a number of combinations of elements of symmetry which may be regarded as *closed conditions*, for if a set has only the three elements of symmetry defined by one of these combinations, it has no other elements of symmetry. These combinations are:

(1) a plane of symmetry, an axis of symmetry perpendicular to the given plane, and the point of intersection of the given axis and plane as a center of symmetry;

(2) an axis of symmetry and two perpendicular planes of symmetry through the given axis;

(3) three mutually perpendicular axes of symmetry:

Raising the restrictions as to perpendicularity on the elements of symmetry in the first part of article III and dealing instead with sets having as elements of symmetry two planes or two axes which intersect in any angle $0° < \alpha < 180°$, we find by a procedure differing only from that used above in that cylindrical co-ordinates are used, that if a set M is symmetric with respect to each of two planes whose respective equations in cylindrical co-ordinates are $\theta = 0$, $0 = \alpha$, then the planes $\theta = m\alpha$ with m any integer, are planes of symmetry of the set M. It follows that if the set has only two planes of symmetry these planes must be perpendicular. Also it can be shown that if $\theta = 0$ and $\theta = \alpha$ are planes of symmetry of a set M, if there exists an integer n such that $n\alpha$ is an odd multiple of $90°$, then the line of intersection of the two planes is an axis of symmetry of M. The sufficient condition of Theorem 3 follows as a corollary of this theorem. Similar theorems dealing with two axes of symmetry may be established.

IV. Theorems concerning sets with parallel or non-coincident elements of symmetry

The first group deals with sets having two elements of symmetry, such that the centers do not lie on the planes or axes and the axes and planes are parallel.

THEOREM 1.—*If a set M has a center of symmetry and an axis of symmetry which does not pass through the center, then*

(*a*) *the set M is unbounded;*

(*b*) *the set has infinitely many centers of symmetry which form an unbounded set lying on a line through the given center and perpendicular to the given axis of symmetry;*

(*c*) *the set M has infinitely many axes of symmetry parallel to the given axis, and lying in the plane of the given center and axis;*

(*d*) *the set of points formed by the intersection of the axes of symmetry and the line of centers is an unbounded set, no point of which is a center of symmetry of the set M.*

Let M have as an axis of symmetry the X-axis, and as a center of symmetry the point A $(0, y_0, 0)$. Choose a point P (a, b, c) of the set and consider first that A is a center and then that the X-axis is an axis of symmetry. We have belonging to M $(-a, 2y_0-b, -c)$, $(-a, -2y_0+b, c)$, $(a, 4y_0-b, -c)$, ... $(-a, \mp[4k+2]\,y_0 \pm b, \pm c)$ and $(a, \mp 4ky_0 \pm b, \pm c)$ where k is any positive integer. Taking the axis first we have the points $(-a, \pm[4k+2]y_0 \pm b, \pm c)$ and $(a, \pm 4ky_0 \pm b, \pm c)$. These points constitute the subset. Since k may have any integral value the set is unbounded.

The points of the subset lie on four lines $x=a$, $z=\pm c$, and $x=-a$, $z=\pm c$.

Since (a, b, c) is a point of M, $(-a, [4k+2]\,y_0-b, -c)$ are points of M. Hence $(0, [2k+1]\,y_0, 0)$ are centers of symmetry for the points on these two lines. Taking the points which lie on the other two lines we that $(0, [2k+1]y_0, 0)$ are centers for these points. Hence these centers are centers of symmetry for the subset. But M is made up of a series of such subsets which lie on lines determined by the co-ordinates of the point chosen (a^i, b^i, c^i). But all these subsets must have the points $(0, [2k+1]\,y_0\, 0)$ as centers of symmetry for the co-ordinates of these points are independent of the points chosen to determine the subsets.

Similarly the set has the axes of symmetry $y=2ky_0$, $z=o$, for the points on the plane $x=a$ have the centers $(a, 2ky_0, 0)$ and the

points on $x=-a$ have the centers $(-a, 2ky_o, 0)$. Therefore, for the other points of the set on planes $x=a^i$, we have the centers $(a^i, 2ky_o, 0)$. But all these points are collinear in $y=2ky_o$, $z=0$. The intersections of these axes of symmetry with the lines of centers of symmetry of the set form an unbounded set, $(0, 2ky_o, 0)$, no point of which is a center of symmetry of the set, for we note that the y-co-ordinates of the centers of symmetry are odd multiples of y_o while the y-co-ordinates of these intersections are even multiples of y_o.

Other theorems concerning sets with either a parallel axis and plane, or a center and an axis, or two centers, or two axes, etc. may be considered. The results of these theorems are summarized as follows: (1) all sets are unbounded; (2) a set having more than one center, more than two parallel axes or more than two parallel planes has, respectively, an infinite number of centers, an infinite number of parallel axes, or an infinite number of parallel planes; (3) a set having two elements of symmetry, each of a different type as a center and a plane, has an infinite number of elements of symmetry of each type.

The following discussion deals with a set having three elements of symmetry which are restricted as above. Consider a set with a center of symmetry and two parallel planes of symmetry.

Let the center be at the origin and the two planes T_1 and T_2 be, respectively, $y=y_1$ and $y=y_2$. To determine the typical subset from a chosen point (a, b, c) of the set consider the elements of symmetry in the order, O, T_1, T_2. We have the points, $(-a, -b, -c)$, $(-a, 2y+b, -c)$, $(-a, -2y+2y-b, -c)$, $(a, 2y_1-2y_2+b, c)$ and $(a, 2y_2-b, c)$. Determine a point symmetric to each of these points with respect to the remaining element of symmetry not used to derive it or to determine the succeeding point. These points are $(-a, 2y_2+b, -c)$, $(a, -2y-b, c)$ $(-a, 4y_1-2y_2+b, -c)$, $(a, 2y_1+4y_2-b, c)$ and $(a, -2y_2+b, -c)$. Now find points symmetric to each of these points with respect to the two elements of symmetry other than the one used to determine the points themselves. For these ten new points repeat this last procedure. Two other collections of points may be determined by taking the elements of symmetry in the orders, T_1 O, T_2, and T_1, T_2, O.

A systematic arrangement of the points shows that they lie on two parallel lines. On the line $x=a$, $z=c$, we have the points whose y-co-ordinates are:

$$2ry_1+2sy_2+b, \quad \text{where } (r+s) \text{ is an even integer;}$$

and

$$2ny_1+2my_2-b, \quad \text{where } (n+m) \text{ is an odd integer.}$$

On the line $x = -a$, $z = -c$, we have the points whose y-co-ordinates are

$$2ny_1 + 2my_2 + b, \quad \text{where } (n+m) \text{ is an odd integer;}$$

and

$$2ry_1 + 2sy_2 - b, \quad \text{where } (r+s) \text{ is an even integer.}$$

The integers may be positive or negative. The set is unbounded.

Pairing the points (a, b, c) and $(-a, 2ry_1 + 2sy_2 - b, -c)$, where $(r+s)$ is an even integer, we find that the points $(0, ry_1 + sy_2, 0)$, where $(r+s)$ is an even integer are centers of symmetry for these two groups of points. By choosing properly our values for n and m in the points $(a, 2ny_1 + 2my_2 - b, +c)$ and $(-a, 2ny_1 + 2my_2 + b, -c)$ we may take them in pairs which are symmetric with respect to the centers $(0, ry_1 + sy_2, 0)$. Since the co-ordinates of these centers are independent of the co-ordinates of (a, b, c), these centers are centers of symmetry of the set.

The points on the lines $x = a$, $z = c$ and $x = -a$, $z = -c$, have the centers $(a, [n+r]y_1 + [m+s]y_2, c)$ and $(-a, [n+r]y_1 + [m+s]y_2, -c)$ where $(r+s)$ is even and $(n+m)$ is odd, on their respective line. Since the y-co-ordinates of these centers are independent of b, all the subsets of M have centers coplanar in $y = (n+r)y_1 + (m+s)y_2$ where $(r+s)$ is even and $(n+m)$ is odd. Hence M has the planes of symmetry $y = k_1y_1 + k_2y_2$ where $(k_1 + k_2)$ is odd.

These planes intersect the line of centers $x = o$, $z = o$ in an infinite set of points $(0, k_1y_1 + k_2y_2, 0)$ where $(k_1 + k_2)$ is odd. These points can be centers of symmetry of the set only when $(r - k_1)y_1 = (k_2 - s_2)y_2$ where $(r+s)$ is an even integer and $(k_1 + k_2)$ is an odd integer. Taking all the possibilities of choosing the integers r, s, k_1 and k_2 to satisfy the above restrictions we may have $(r-k)y_1 = (k_2 - s)y_2$ only if we have $(2h+1)y_1 = 2qy_2$ or $2qy_1 = (2h+1)y_2$. Hence we have,

THEOREM 2. *If a set M has two parallel planes of symmetry and a center of symmetry not contained in either plane, then*

(*a*) *the set M is unbounded;*

(*b*) *the set has an infinite number of centers of symmetry on a line through the given center and perpendicular to the given planes;*

(*c*) *the set has an infinite number of planes of symmetry parallel to the given planes of symmetry;*

(*d*) *in general, the planes of symmetry intersect the line of centers in an infinite set of points, no point of which can be a center of symmetry.*

A NOTE ON THE FUNCTION

$$\sigma(z) = \prod_{n=1}^{\infty} (1 - z^2/\lambda_n{}^2).$$

BY

V. GANAPATHY IYER

1. Let $[\lambda_n]$, $0 < \lambda_1 < \lambda_2 \ldots < \lambda_n \rightarrow \infty$, be a measurable sequence of order one and density D, that is,

$$\lim_{r \to \infty} \frac{n(r)}{r} = \mathrm{D}, \qquad \ldots \quad (1)$$

where $n(r)$ is the number of numbers λ_n not exceeding r. Let

$$\sigma(z) = \prod_{n=1}^{\infty} (1 - z^2/\lambda_n{}^2) \qquad \ldots \quad (2)$$

Then

$$\lim_{r \to \infty} \frac{\log |\sigma(re^{i\theta})|}{r} = \pi \mathrm{D} |\sin \theta|, \qquad \ldots \quad (3)$$

uniformly in any angle not containing $\theta = 0$ or π.

1·1. The result embodied in relation (3) is important in many investigations [1]. Bernstein [2] has deduced (3) from some general properties of (2) which are adapted to his investigations on Dirichlet's series. Therefore, it is worthwhile recording a simple and direct proof of (3). In this note, I give such a proof.

2. *Proof of the relation* (3): We have,

$$\log |\sigma(re^{i\theta})| = \sum_{n=1}^{\infty} \log \left| 1 - \frac{r^2}{\lambda_n^2} e^{2i\theta} \right|$$

$$= \tfrac{1}{2} \sum_{n=1}^{\infty} \log \left(1 - \frac{r^2}{\lambda^2_n} \cos 2\theta + \frac{r^4}{\lambda_n^4}\right)$$

$$= \tfrac{1}{2} \int_0^{\infty} \log \left(1 - \frac{r^2}{t^2} \cos 2\theta + \frac{r^4}{t^4}\right) dn(t), \qquad \ldots \quad (4)$$

for all $r \geqslant 0$ provided $\theta \neq 0$ or π since in this case the expression under the logarithm in (4) does not vanish for any r or t. Integrating

(1) See, for instance, my papers in the *Quart. Jour. Math.* (Oxford Series), 8 (1937), pp. 103-106 and pp. 131-141. Also, A. Pfluger, *Compositio Mathematica*, 4 (1937), pp. 367-372.

(2) V. Bernstein, "*Series de Dirichlet*", Borel Tracts (1933), pp. 279 81. Polya has also given a proof [*Math. Zeit.*, 29 (1929), pp. 549-640] but I do not know the nature of this proof.

(4) by parts and noting that $n(r)/r^\rho \to 0$ as $r \to \infty$ for $\rho > 1$ in virtue of (1), we get

$$\log|\sigma(re^{i\theta})| = \int_0^\infty \frac{r^2(r^2-t^2\cos 2\theta)}{r^4-2r^2t^2\cos 2\theta+t^4}\frac{n(t)}{t}\,dt$$

$$= r\int_0^\infty \frac{N(rv)(1-v^2\cos 2\theta)}{v^4-v^2\cos 2\theta+1}\,dv \qquad \dots (5)$$

on putting $t=rv$, $n(t)=t\,N(t)$. Splitting $(0,\infty)$ into $(0, 1/\sqrt{r})$ and $(1/\sqrt{r},\infty)$ and noting that $N(rv)\to D$ as $(rv)\to\infty$, we get from (5) that, uniformly in any angle not containing $\theta=0$ or π,

$$\lim_{r\to\infty}\frac{\log|\sigma(re^{i\theta})|}{r} = D\int_0^\infty \frac{1-v^2\cos 2\theta}{v^4-v^2\cos 2\theta+1}\,dv \qquad \dots (6)$$

The integral on the right side of (6) can be evaluated directly but an indirect argument will give the value more easily. It may be noted that the right side of (6) is independent of the function $\sigma(z)$ so long as $[\lambda_n]$ satisfies (1). Taking $\lambda_n = n/D$, we get $\sigma(z)=\sin \pi Dz$. But, as is easily seen,

$$\lim_{r\to\infty}\frac{\log|\sin(\pi D\,re^{i\theta})|}{r} = \pi D|\sin\theta| \qquad \dots (7)$$

provided $\theta \neq 0$ or π. From (6) and (7) we get (3).

GLEANINGS

65. The more we know the less we know: the less we know of the basic values of life and of character, which are religious, often unreasonable, and in the last analysis beyond reason. The mind coupled to religion, is a stronger mind for it, a mind not so easily swayed by the passions that parade as reason under an enlarged vocabulary. But in deifying the mind we have abandoned God. We have become the fools of reason and the dupes of Scientific truth. As Aldous Huxley has said it, "Ye shall know the truth and the truth shall make ye mad."

DR. HENRY C. LINK in
The Return to Religion.

REFLECTION OF A POINT IN A LINE

By

HUKUM CHAND, *Sanatana Dharma College, Lahore.*

1. *Introduction.*—The reflection of a point P in a line p is usually defined as a point Q such that PQ is bisected at right angles by p. The following is a projective definition. Let the Absolute be a degenerate conic consisting of the points I and J. The lines PI and PJ meet the axis of reflection p in points A and B respectively. The lines AJ and BI intersect in a point Q which is the "reflection" of P in p. The construction shows that the relation between P and Q is mutual. The line PQ is "perpendicular" to p. The point of intersection R of p with PQ is called the "foot of the perpendicular from P (or Q) on p" and R is the "mid-point" of the segment PQ. Now, given P and Q the lines PI and QJ, PJ and QI intersect in points A and B respectively, then Q (or P) is reflection of P (Q) in AB.

In paras 2 and 4 are found the co-ordinates of Q and R. The condition of perpendicularity of p and PQ is determined in para 3. The co-ordinates of R in terms of the co-ordinates of P and Q are found in para 5. A generalization of these ideas is indicated in para 6 where the Absolute is taken to be a non-degenerate conic.

2. Let P (ξ, η, ζ) be the given point and Q $(\xi' \; \eta' \; \zeta')$ its reflection in the line $px+qy+rz=o$. Suppose

$$\Sigma al^2 + 2\Sigma fmn = 0$$

is the line-equation of the Absolute, consisting of the points I and J, in the system of co-ordinates chosen. Thus

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

We denote by capital letters the co-factors of the corresponding small letters in Δ. The equation of the line at infinity IJ can be written in the form

$$\mathrm{A}x + \mathrm{H}y + \mathrm{G}z = 0$$

or two other equivalent forms. Let

$$\Omega \equiv \sum ap^2 + 2\sum fqr \neq 0 \quad \Omega' \equiv \sum ap'^2 + 2\sum fq'r' \neq 0.$$

If the co-ordinates of A and B be respectively (a_1, b_1, c_1) (a_2, b_2, c_2) so that $\sum pa_1 = 0 \quad \sum pa_2 = 0$, the equation

$$(\sum l\xi)(\sum l\xi') + \lambda(\sum la_1)(\sum la_2) = 0$$

represents a system of conics touching the lines PA, PB, QA, QB.

The system includes the Absolute. Consequently

$$\frac{\xi\xi'+\lambda a_1 a_2}{a}=\frac{\eta\eta'+\lambda b_1 b_2}{b}=\frac{\zeta\zeta'+\lambda c_1 c_2}{c}=\frac{\eta\zeta'+\eta'\zeta+\lambda(b_1 c_2+b_2 c_1)}{2f}$$
$$=\frac{\zeta\xi'+\zeta'\xi+\lambda(a_1 c_2+a_2 c_1)}{2g}=\frac{\xi\eta'+\xi'\eta+\lambda(a_1 b_2+a_2 b_1)}{2h}$$

and each one of these ratios equals

$$\frac{\xi(\sum p\xi')+(\xi'\sum p\xi)}{\Omega_1}=\frac{\eta(\sum p\xi')+\eta'(\sum p\xi)}{\Omega_2}=\frac{\zeta(\sum p\xi')+\zeta'(\sum p\xi)}{\Omega_3}=\frac{(\sum p\xi)(\sum p\xi')}{\Omega}$$

where $\Omega_1=\partial\Omega/\partial p\,;\ \Omega_2=\partial\Omega/\partial q\,;\ \Omega_3=\partial\Omega/\partial r$ ... (2 · 1)

Hence we get

$$\xi' : \eta' : \zeta' = \Omega_1\sum p\xi-\Omega\xi : \Omega_2 \sum p\xi-\Omega\eta : \Omega_3\sum p\xi-\Omega\zeta \qquad \dots \quad (2\cdot 2)$$

3. The equation of the line PQ is

$$x(\eta\Omega_3-\zeta\Omega_2)+y(\zeta\Omega_1-\xi\Omega_3)+z(\xi\Omega_2-\eta\Omega_1)=0.$$

Calling this line $p'x+q'y+r'z=0$, we find the condition of perpendicularity, *viz.*

$$p'\,\partial\Omega/\partial p+q'\,\partial\Omega/\partial q+r'\,\partial\Omega/\partial r=0.$$

The relation of perpendicularity is mutual, since

$$\sum p'\,\partial\Omega/\partial p=\sum p\,\partial\Omega'/\partial p'.$$

4*. The foot of the perpendicular from P on the line $\sum px=0$ is the intersection of the line PQ and $\sum px=0$ and so its co-ordinates are given by the equations

$$x:y:z=r\,(\zeta\Omega_1-\xi\Omega_3)-q(\xi\Omega_2-\eta\Omega_1)$$
$$:\ p\,(\xi\Omega_2-\eta\Omega_1)-r(\eta\Omega_3-\zeta\Omega_2)$$
$$:\ q\,(\eta\Omega_3-\zeta\Omega_2)-p(\zeta\Omega_1-\xi\Omega_3)$$

which in virtue of Euler's Theorem on homogeneous function take the form

$$x:y:z=\Omega_1\sum p\xi-2\Omega\xi:\Omega_2\sum p\xi-2\Omega\eta:\Omega_3\sum p\xi-2\Omega\zeta \qquad \dots \quad (4\cdot 1)$$

5. If the points P and Q be supposed to be given, the point (x, y, z) found above defines the mid-point of PQ. We may, therefore, express the co-ordinates of this point in terms of (ξ, η, ζ), (ξ', η', ζ') independent of p, q, r.

The co-ordinates of this point, in virtue of relations (2·1) can be written in the form

$$x:y:z=\frac{\xi}{\sum p\xi}-\frac{\xi'}{\sum p\xi'}:\frac{\eta}{\sum p\xi}-\frac{\eta'}{\sum p\xi'}:\frac{\zeta}{\sum p\xi}-\frac{\zeta'}{\sum p\xi'} \qquad \dots \quad (5\cdot 1)$$

It is found on using relations (2·1) that the co-ordinates of R are

$$\left(\frac{\xi}{\sum A\xi}+\frac{\xi'}{\sum A\xi'},\ \frac{\eta}{\sum A\xi}+\frac{\eta'}{\sum A\xi'},\ \frac{\zeta}{\sum A\xi}+\frac{\zeta'}{\sum A\xi'}\right) \qquad \dots \quad (5\cdot 2)$$

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* The Algebra in this section can be slightly simplified by considering that co-ordinate of R are of the form $(\xi+\rho\xi', \eta+\rho\zeta', \zeta+\rho\zeta')$ and it lies on the line $\sum px=0$.

and the equation of the axis is

$$\begin{vmatrix} a & b & g & \xi \\ h & b & f & \eta \\ g & f & c & \zeta \\ x & y & z & o \end{vmatrix} \frac{1}{A\xi+H\eta+G\zeta} - \begin{vmatrix} a & h & g & \xi' \\ h & b & f & \eta' \\ g & f & c & \zeta' \\ x & y & z & o \end{vmatrix} \frac{1}{A\xi'+H\eta'+G\zeta'} = 0.$$

6. We may now consider the problem when the Absolute is a proper conic. Let P be a point and p a line not passing through P. Suppose that the Absolute is a non-degenerate conic $\sum$ not tangent to p. The tangents through P to $\sum$ meet p in A and B. Let the other tangents from A and B to $\sum$ meet in a point Q which I call the "reflection" of P. The lines p and PQ are called "perpendicular" since they are conjugate w.r.t. Ω, while the intersection R of p and PQ is called the "foot of the perpendicular" from P (or Q) on p and also a "mid-point of PQ". Conversely, given the points P and Q, the four tangents through P and Q meet in three pairs of points P, Q; A, B; A′, B′. The line PQ meets AB and A′B′ in R and R′ and R and R′ are called the mid-points and the two lines AB, A′B′ are axes of reflection.

In passing, it may be pointed that the results obtained in paras 2, 3, 4 hold good in the present case when $\Delta \neq 0$. We find from (2·1)

$$4\Delta p = \frac{\Omega}{\sum p\xi}\frac{\partial s}{\partial \xi} + \frac{\Omega}{\sum p\xi'}\frac{\partial s'}{\partial \xi'}$$

$$4\Delta q = \frac{\Omega}{\sum p\xi}\frac{\partial s}{\partial \eta} + \frac{\Omega}{\sum p\xi'}\frac{\partial s'}{\partial \eta'}$$

$$4\Delta r = \frac{\Omega}{\sum p\xi}\frac{\partial s}{\partial \zeta} + \frac{\Omega}{\sum p\xi'}\frac{\partial s'}{\partial \zeta'}$$

where $s \equiv \sum A\xi^2 + 2\sum F\eta\zeta, \qquad s' \equiv \sum A\xi'^2 + 2\sum F\eta'\zeta'.$

Hence

$$4\Delta = \frac{2\Omega s}{(\sum p\xi)^2} + \frac{\Omega}{\sum p\xi \sum p\xi'} \sum \xi \frac{\partial s'}{\partial \xi'}$$

$$= \frac{\Omega}{(\sum p\xi)(\sum p\xi')} \sum \xi' \frac{\partial s}{\partial \xi} + \frac{2\Omega s'}{(\sum p\xi')^2}$$

whence $s : (\sum p\xi)^2 = s' : (\sum p\xi)^2.$

We thus get two equal and opposite values of the ratio $(\sum p\xi)/(\sum p\xi')$ which correspond to R and R′. These points form with P, Q a harmonic range. Thus the co-ordinates of R and R′ are

$$\left(\frac{\xi}{\sqrt{s}} \pm \frac{\xi'}{\sqrt{s'}},\ \frac{\eta}{\sqrt{s}} \pm \frac{\eta'}{\sqrt{s'}},\ \frac{\zeta}{\sqrt{s}} \pm \frac{\zeta'}{\sqrt{s'}}\right).$$

The axes of reflection $\sum px = 0$ will be lines that pass through R and R′ and their equations will be found to be

$$\frac{1}{\sqrt{s}} \sum x \frac{\partial s}{\partial \xi} \mp \frac{1}{\sqrt{s'}} \sum x \frac{\partial s'}{\partial \xi'} = 0.$$

NOTES AND DISCUSSIONS

Teaching of Mathematics in German Schools

PROF. DR. HELMUT HASSE, GOTTINGEN,
Mathematisches Institut der Universität 9-8-1937

To

A. NARASINGA RAO,

Editor of The Mathematics Student,

ANNAMALAINAGAR.

DEAR SIR,

I have been reading Dr. S. Mukhopadhyaya's note in vol. IV, No. 4 of your periodical. I feel it my duty to let you know that several of the author's statements about German Schools do not agree with the facts.

There are 3 stages in our educational system :

Vorschule	(6—9)	(Elementary School)
Hohere Schule	(10—18)	(High School)
Hoch Schule		(University)

The author uses the term 'Hoch Schule' for what we call 'Höhere Schule'. He also forgets to allow for the elementary course in the Vorschule that does not belong to the Höhere Schule.

Classes from 8-11 and 2-5 may be held in a very few exceptional cases. As a rule classes are held from 8-1 or 1-30, and only sports, drawing, music in the afternoon. Recently afternoon classes have been abolished altogether. There are usually 6 classes from 8—1·30, each lasting for 45 minutes, with 4 intervals of 10 minutes and one of 20 minutes. The latter gives the students an opportunity for eating the so-called second breakfast, consisting of bread and butter, prepared beforehand at home, and often a glass of milk on sale in the school premises. This is essential, since our German first breakfast, taken immediately after rising, is by far not so substantial as the average English breakfast. There is no meat and no warm dish with it, but only bread and butter, some jam, and milk or coffee.

Almost all German schools are day-schools, as the author rightly points out, although there are more boarding schools than the one the

author mentioned, for example the famous boarding school at Schulpforta with tradition of several centuries, where special consideration is given to classics. Boys must write a Latin essay there for the final examination, and be able to carry on a scientific conversation in Latin. On the whole the special study of classics is gradually decreasing in Germany. There are only a few schools of the old *Gymnasium* type left now. The overwhelming majority are of the two other types: *Realgymnasium* (only Latin, no Greek), and *Oberreal schule* (no classics at all). A recent reform intends uniformising even the two latter types. In a few years the average German Higher School will have English as the main foreign language, taught from 10-18 years, and French from 14-18 years only, whereas formerly in most of our schools either Latin or French was the main foreign language.

Schools with an own swimming bath are rare exceptions. Athletics classes are held in the municipal swimming baths, though, occasionally, in particular during summer, when open air swimming is possible.

Gottingen, 9-8-1937.

Sincerely yours,
H. HASSE

Advice to a young researcher

MY DEAR (RESEARCH) FELLOW,

1. You write to me that you have taken a high Honours Degree and that you are keen on a serious study of Mathematics with a view to research therein; and you wish me to advise you as to how you should set about it. You are a member of a class of young enthusiasts happily on the increase in our country, who are in an exactly similar situation, and it will be but in the spirit of Mathematics if through this reply in the STUDENT I deal not only with you individually but also with the whole class.

2. Firstly, it is good to have high ideals. Resolve that you will be not a mere dabbler in research, but that you will seek to make contributions to the subject which will be recognised as valuable. Your success may be complete or partial, but you will have the satisfaction of having made an honest attempt. As the *Gita* has it "You have the right to the task, but not to the fruits thereof".

I consider a sound physique as an invaluable foundation on which the superstructure of research should be erected. Keep your body in efficient condition by means of exercise, healthy habits and controlled

living. The pursuit of Truth in whatever field calls for moral qualities of a high order such as Courage, Intellectual Honesty, Humility and the faculty of Self Criticism. Unless you have these your work will lack the proper tone and will be of doubtful quality. All these will be of no use unless you are willing to work hard; and if you combine with them a sense of proportion and the saving gift of humour, you are indeed very well equipped morally for the task.

3. Language Equipment

You must seek to obtain, if you do not have it already, a sufficient mastery of English to be able to express yourself with vigour and flexibility. The equipment of the average Honours graduate is somewhat below this standard, but you can easily make up by intelligent reading of suitable material. Besides English, you should seek to have a working knowledge of both French and German, or at least of one of them. German will probably be the more useful of the two but French is, in my opinion, somewhat easier to learn so that there is not much to choose between them. The vocabulary required for following mathematical papers is not extensive, but you should pick up a fair amount of grammar; otherwise you will learn just enough to misunderstand the author! The best plan is to join a regular course, but even if you cannot do so, you can acquire the knowledge you need by self study in a year or less. If you have a flair for languages you may add Italian to your collection and it will come in handy.

4. Preliminary Browsing

You cannot begin your (higher) mathematical education better than by studying something of the History of Mathematics with a view to acquainting yourself with the major movements in the subject and the classical treatises in the various topics thereof. Prof. G. A. Miller's *Historical Introduction to Mathematical Literature* should do excellently for the purpose, and you may also consult Cajori's *History of Mathematics.* You may also browse through the Presidential Addresses delivered at mathematical conferences and public addresses at International Mathematical Congresses giving a bird's-eye-view of the developments of various topics and suggesting problems for investigation. Such review articles will also be found in the *Jahresbericht der deutsche Mathematiker-Vereinigung.*

It is not a bad idea to get seized, fairly early, of a specific problem for investigation neither trivial nor too ambitious. Such a course will give a definite orientation and an added interest to the

studies it suggests. After solving it one way, try to look at it from various other points of view bringing fresh ideas and outlook, and carrying on such further basic studies as may be required for the purpose. Try to generalise the result in various directions and see if such generalisation could be pushed still further by a refinement of your methods. Consult the literature on the subject available in reference books like

Encyklopädie der mathematischen Wissenschaften
Pascal: Repertorium der höheren Mathematik
Weber and Wellstein: Encyklopädie der elementar Mathematik

The following series of publications are important to the advanced worker and you should consult those books of the series which deal with the topic or any cognate subject. They will give information as up-to-date as was available at the time of publication:

Memorial des Sciences Mathematiques
Monographes sur la Theorie des Foncions
Actualités Scientifiques et Industrielles
Die Grundlehren der mathematischen Wissenschaften
Ergebnisse der Mathematik und ihrer Grenzgebeite
American mathematical Society's Colloqium publications
Cambridge Tracts in mathematics and mathematical physics

In my own experience I have found some of the booklets of the series *Teubner's mathematische Leitfaden* though small, very useful and suggestive.

5. Periodical Literature

When you have chosen your favourite field of study and learnt the main results therein, it is time to keep in touch with current developments by reading such papers as may be appearing from time to time. Summaries of such articles, classified according to subjects, appear in

Zentralblatt für Mathematik und ihre Grenzgebeite
Jahrbuch über die Fortschritte der Mathematik
Revue Semestrielle des publications Mathematiques

It is also desirable that you should make it a habit to look into the issues of the various mathematical journals as they are published. In the beginning *The Mathematical Gazettee* and I hope also, *The Mathematics Student* should be of help in widening your range and stimulating interest. After a time you should be able to study with profit the original papers relating to your field of study appearing in high class mathematical journals among which I will include among foreign publications

Acta Mathematica
Transactions of the American Mathematical Society
Annals of Mathematics
Quarterly Journal of Mathematics
Proc. of the London Mathematical Society
Mathematische Annalen
Mathematische Zeitschrift
Crelle's Journal
Bulletin de la Société Mathematique de France
Annales de l'Ecole Normale Supérieure
Rendiconti del circolo matématic di Palermo.

and among those published in India

The Journal of the Indian Mathematical Society
Bulletin of the Calcutta Mathematical Society

The above list is by no means exhaustive and does not take into account the various Italian *Atti*, the Japanese Journals, and periodicals like Comptes Rendus, Gottinger Nachrichten, Proc. of the Camb. Phil. Soc. and of the Royal Society, and Publications of the various Indian Academies which deal with several subjects including mathematics. Besides these there are journals like Acta Arithmetica, Fundamenta Mathematica, etc., which specialise in particular branches of Mathematics. If a University library be accessible, there should be no difficulty in getting these periodicals. To help those not having these facilities some Mathematical Societies, in particular the Indian Mathematical Society, arrange for the circulation of current periodicals. If you are regular in returning books and journals in time, you may enjoy the privilege of having them in circulation at the small additional expenditure of paying the transit charges.

You should also seek to keep in touch with others interested in the same or cognate fields by correspondence and through personal contact. The various mathematical and scientific conferences which meet in India offer splendid opportunities for forming and maintaining such contacts.

6. Publication of Papers

When you have carried through an investigation as far as you can usefully pursue it, settled all obscure points and have clear notions in your own mind, it is time to write it out for publication. Make it brief and telling, and rewrite it half a dozen times, if necessary, till it has the proper tone and form. A mathematical paper is a work

of art, albeit in an unusual medium, and should have perfect proportion of its parts, its high lights and its shadows, and should breathe vigour and confidence. For this you should omit or simply indicate mechanical work like simplifications, substitutions and show by your language and the amount of space you devote to each part, the importance you attach to it. State quietly; the results you regard as your special contribution say that you believe them to be new and do not shout from the house-tops as if you had discovered the secrets of the universe. Pick your words carefully so that they convey just what you intend; for instance if a result appears almost certain but you have not verified it, say it would appear that such a result must be true.

Study carefully the notation and the symbolism used in the best papers so that the final printing may be attractive. Write a neat hand and let your formulae be unambiguous and your figures well drawn and you will earn the gratitude of the overworked editor and the dumb compositor.

When you have written a number of papers, you will have arrived at the stage where you do not need advice from others, and I wish you Godspeed in your future endeavours.

I remain,
Ever (mathematically) yours,
A. NARASINGA RAO.

Annamalainagar,
29th November, 1937.

An example of a finite set satisfying only the first three postulates of the group theory.

It is well known that a set of elements forms a group with respect to a given law of combination, if the following four postulates are satisfied.

1. the set must be closed (i.e.) if A and B are any two elements of the set, $A \cdot B$ also belongs to it;

2. the assosciative law is true;

3. there is one and only one element I so that for any element A, $AI = IA = A$

4. To every element A corresponds an element A^{-1} so that $AA^{-1} = I$.

In text-books on groups, the usual example given, of a set obeying only the first three postulates, is the set of positive integers including

zero, the law of combination being addition. To a student of transfinite numbers, the following example of a set satisfying only the first three postulates seems to be ready at hand. Consider the transfinite numbers Aleph Zero $\aleph_0$. (the cardinal number of an ennumerable aggregate), and C that of the continnum. With respect to multiplication the set behaves as follows

$$\aleph_0 \times \aleph_0 = \aleph_0. \quad \aleph_0 \times C = C \times \aleph_0 = C. \quad C \times C = C.$$

It is evident from the above that they satisfy the first three postulates, $\aleph_0$ being the identity. But they do not satisfy the fourth postulate, since C has no inverse. Hence the elements $\aleph$ and C do not form a group.

Annamalainagar. B. RAMAMURTI.

GLEANINGS

1,000,000,000

66. "It has been only a little over a billion (10^9) minutes since Christ was born in Bethlehem.

If some man had spent a thousand dollars the day that Christ was born and every day thereafter as long as he lived, and if one of his descendants had continued to dispense funds at the same rate ever since, it would be some eight hundred years yet before a billion dollars would be spent—that is, around the year 2736.

Somebody announced that his calculations showed that it is only a little over a billion and a half inches around the world at the equator, and but slightly more than a billion and a quarter feet to the moon.

To change the picture, we found that a man whose pulse is around eighty to the minute will be almost twenty-four years old before his heart beats amount to a billion.

It is a rapid speaker who can enunciate a hundred and fifty words to the minute. But if one could do that and hold out at the same rate for eight hours a day, month after month, indefinitely, it would take him almost thirty-eight years to deliver a billion words, with no time out for Sundays, holidays, or vacation.

A good ear of corn has about a thousand kernels. But it would require some twenty thousand bushels, or a matter of five hundred tons of shelled corn to count out a billion kernels.

And if a man should set out to count the kernels in that twenty thousand bushels, and could maintain a rate of hundred to the minute for eight hours a day, it would take him a matter of fifty-seven years to finish the job.

In dollar bills laid end to end, one billion would reach one million, one hundred and sixty-seven thousand miles. This is more than forty-six times around the earth, and enough to reach to the moon and back twice, with some two hundred thousand miles to spare.

Extract from a reprint in
The Mathematics Teacher, *Oct. 1937.*

[library stamp]

ASTRONOMICAL NOTES

For December 1937 & January 1938

Sun. Between the first of December and the end of January 1938 the Sun will be moving from the centre of the constellation Scorpio and passing Sagittarius will reach the central part of Capricornus. On 22nd December the Sun enters the winter solstice and on this day the length of the day is shortest and that of the night longest.

The position of the Sun on the 1st of December and January will be as follows for Trivandram Mean Noon.

	R. A.	Decln.
1st December 1937.	16 h 28 m	21° 44′ South
1st January 1938.	18 h 45 m	23° 4′ South

Moon. The following phenomena of the Moon will occur during the months of December and January.

	December 1937.		January 1938.	
New Moon	3rd at 4 41 A. M.	I. S. T.	2nd at 0 28 A. M.	I. S. T.
			31st at 7 5 P. M.	,,
First Quarter	11th at 6 42 A. M.	,,	9th at 7 43 P. M.	,,
Full Moon	18th at 0 22 A. M.	,,	16th at 11 23 A. M.	,,
Last Quarter	24th at 7 50 P. M.	,,	23rd at 1 39 P. M.	,,
Apogee	3rd at 10 30 P. M.	,,	27th at 11 30 A. M.	,,
	30th 11 30 P. M.	,,		
Perigee	17th at 7 30 P. M.	,,	15th at 7 30 A. M.	,,

Mercury will be in the constellation Sagittarius during December and January. During the early part of December the planet will be an evening object and by the end of the month it will be too near the sun to be visible; by the end of January it will appear as a morning star.

The following phenomena of Mercury will occur during December and January.

	December 1937	January 1938
Conjunction with Moon	4th	29th
Stationary	20th	9th
Greatest elongation	12th	20th
Inferior conjunction with Sun	30th	

The position of Mercury will be as follows

	R. A.	Decln.
1st December 1937.	17 h 43 m	25° 45′ South
1st January 1938.	18 h 22 m	20° 25′ South

Venus. On 1st December Venus enters Scorpio and passing Sagittarius enters Capricornus on the 17th January and remains in that constellation till the end of January. During December the planet will be visible as a morning object, and during the latter part of January it will be too near the Sun to be visible. Venus will be in conjunction with the Moon on 1st December.

The position of the planet will be as follows.

	R. A.	Decln.
1st December 1937	15 h 24 m	17° 31′ South
1st January 1938	18 h 9 m	23° 33′ South

Mars. During the 1st week of December, Mars will be in Capricornus, and on the 12th December it enters Aquarius and passing this constellation, enters Pisces on 20th January. During these months Mars is an evening object seen towards the West of the meridian. Mars will be in conjunction with the Moon on 9th December and 7th January. Its position will be as follows.

	R. A.	Decln.
1st December 1937	21 h 10 m	17° 51′ South
1st January 1938	22 h 40 m	9° 25′ South

Jupiter. Jupiter will be in Capricornus throughout the period. Like Mars this planet is an evening object and is visible in the neighbourhood of Mars and west of it. Towards the end of January the planet will be too near the Sun to be visible. Jupiter will be in conjunction with the Moon on 7th December and 3rd January.

Its position will be as follows.

	R. A.	Decln.
1st December 1937	19 h 53 m	21° 25′ South
1st January 1938	20 h 21 m	20° 4′ South

Saturn. This planet will be in Pisces throughout December and January and will be an evening object to be seen through the telescope. On December 2nd the planet will be stationary and on 12th December and 8th January it will be in conjunction with the Moon.

The position of the planet will be as follows :—

	R. A.	Decln.
1st December 1937	23 h 58 m	2° 54′ South
1st January 1938	0 h 1 m	2° 28′ South

Uranus. This planet is visible in the night with the aid of a powerful telescope. It will be stationary on the 18th January and after that date the planet's motion is regular.

The positions of Uranus will be as follows :—

	R. A.	Decln.
1st December 1937	2 h 33 m	14° 37′ North
1st January 1938	2 h 30 m	14° 22′ North

Neptune. Unlike the planet Uranus the motion of this planet during December is direct and on December 24th it will be stationary and its motion is retrograde after this date.

The position of the planet will be as follows :—

	R. A.	Decln.
1st December 1937	11 h 29 m	4° 34′ North
1st January 1938	11 h 29 m	4° 32′ North

Jupiter's atmosphere. In a paper contributed by B. M. Peek to the *Monthly Notices of the Royal Astronomical Society* Vol. 97 No. 8 the relations between depth, pressure, temperature and density in Jupiter atmosphere are investigated by considering three models.

1. The adiabatic,
2. The isothermal,
3. A compromise model in which a relation is assumed between the depth and the lapse rate.

A New Comet. Mr. Finsler of Zurich discovered on 4th July a new comet of the 7th magnitude in the constellation Perseus. It has been observed that this comet has a nucleus and a short tail of about one degree in length. The daily motion of the comet was towards North and East and from computation of its orbit it was found that the comet will be in perihelion on 12th August.

Maintenance of the pulsation in Cepheid Variables. The possibility of the existence of a self regulating mechanism maintaining the pulsation of a cepheid variable is discussed by Dr. J. Woltjer Jun in a letter to *Nature* of July 1937. It depends upon the superposition

of the oscillations of two modes of vibrations one of which is free and the other forced.

• **Masses and parallaxes of Binary Stars.** A Statistical determination of the masses and parallaxes of 124 Binaries (whose orbits have been computed by Prof. R. K. Young) has been made by Dr. C. Barnes in the *Monthly Notices of the Royal Astronomical Society*, Vol. 97. He has given the mean values of $\rho\omega^2$ and $[\rho\omega^2]^{\frac{1}{2}}$ for orbits of given eccentricity. These are quantities appearing in the equations for determining the combined mass and the parallax.

Double Stars and Relativity. At the Tercentenary conference of Arts and Sciences at Harvard in September 4, 1936, Prof. T. Levi Civita has shown that the relativistic equations of motion of two bodies of comparable size can be integrated by treating the relativity effects as perturbations. The following two results have been obtained (1) A formula for the angular precession of the periastron of the satellite Star. (2) The acceleration of the centre of mass of the double star lies in the plane of the relative orbit. This has a fluctuating period and a secular part accumulating in effect during successive revolutions. The increase of the velocity of the centre of mass in a century is given, a result which could be verified by spectroscopic observations.

Explanation for the variation of the brightness of the earth-shine on the moon.

(1) We look the earth-lit portion always along the path of the rays whereas the Sun's rays fall on the moon at all angles relative to us. At full moon the Sun's rays come from behind the earth; but near new moon at oblique angles from behind the moon and then every roughness, whether mountain or rock has its shadowy side towards us. These diminish the average brightness of the sun-lit portion near new moon. Hence the earth's shine is relatively brighter when the moon is crescent.

(2) Another cause is that the amount of light falling on the moon from the earth is decreasing as the moon grows older and older. The earth-shine at half moon is only 1/3 as bright as what is observed when the moon's elongation is 30°. The earth's reflecting power is much more than that of the moon and varies according to the atmospheric condition, clouds, etc. This is 30% higher in February than in August. The Earth at full phase would send to the moon 1/9000 part of the sun's light and would look bluish in

colour as the Earth sends more blue light due to the atmosphere scattering more blue light.

Globular Corona. (*Nature* 2—10—37) Coronal photographs secured from the stratosphere at the total eclipse of the Sun in June last, show that the successive isophotos of the corona round the Sun are nearly circular. Earlier workers in the field have shown the difference in the behaviour during the solar cycle of this more regular solar corona and the super-posed streamers. Investigation of the polarization in two regions (photographic and visual) of the solar spectrum was also made by the observers from Abastrmain Observatory.

The conclusions reached are :—

(1) The degree of polarization is different for different regions of the spectrum.

(2) The change of the polarization with increasing distance from the moon's limb is different for different heliographic latitudes and depends on the structure of the solar corona.

(3) The direction of polarization undergoes a regular perturbation in the coronal streams.

Distribution of matter in Space (*Nature* 27—9—37). Mr. Reynolds in an article in the September issue of *Nature* has come to the conclusion that though uniform distribution of matter in space is a useful assumption, it is not a fact supported by observation. The evidence supplied by the nearer systems of nebulae within a radius of 4000 Kiloparsecs is distinctly against the assumption, while beyond this limit there is not sufficient observational material to arrive at a conclusion in its favour.

Trivandrum.

H. SUBRAMANI AIYAR,
Govt. Astronomer.

REVIEWS

SIR JOSEPH LARMOR, (Edited by)

Origins of Clerk Maxwell's Electric ideas as described in familiar letters to William Thomson, (Cambridge University Press, 1937, pp. 1-56).

The publishers of this collection of letters from one scientific man of genius to another genius have laid a large circle of readers under a debt of gratitude. It is perhaps true that the letters relate to an earlier epoch of Maxwell's career when he had not yet reached maturity as regards his electric ideas. It is really difficult for one making a careful study of these letters to say how exactly his ideas crystallised and culminated in the derivation of the famous equations of the electro-magnetic field. The letters on electric topics are so thoroughly mixed up with others of a different nature that it is hard to find any sort of logical sequence in the ideas. Many readers who go through the book would very much wish to know what replies Kelvin gave to these letters.

As a human document the letters are of the greatest interest. It brings forth clearly the varied scientific interests of Maxwell and his clear methods of thinking and investigation. The great pains he took to devise new and useful methods in teaching indeed come to us as a revelation. His letter on "Peacocks as Gardeners" show that even in cultivating the hobby of gardening he studied all the details with care. Nowhere does one find in the letters the slightest touch of idiosyncracy usually attributed to men of genius—unless one were to include under the category his signing himself as dp/dt! Ability in experimental manipulation and theoretical investigation seem to have been very happily blended in him, and is in shining contrast to the state of affairs in Physics at the present day. Thus one finds a letter on the theory of bending of surfaces is followed by another on the construction of an Eye-Speculum on Helmholtz's Principle followed again by an investigation of the stability of fluid motion—a remarkable combination of pure mathematics, experimental physics and theoretical physics.

In one of his letters Maxwell asks for information regarding the correction of exercises of his students and wants to know about the method of making one man look over another's exercise. One wonders if Maxwell actually put this method into practice in his classes. If he did so the following quotation taken out from the Maxwell commemoration volume becomes quite understandable—

"Each term he gave a course of lectures which did not attract as large an audience as they deserved";

any way a great consolation to us all in the teaching profession!

B. S. M.

KATSH, Abraham I: *Einstein's Theory of Relativity*, Tel-Aviv, 1936. Pp. 80

This is a book in Hebrew from that new centre of intellectual activity, the Jewish University at Tel-Aviv, Palestine, which seeks to explain in simple language the more salient features of the Theory of Relativity. Mathematical

formulae (relating to the Special Theory) are reduced to a minimum and relegated to the appendix. As probably the first book of its kind, it should be welcome to Hebrew readers for getting an idea of the subject though it cannot be expected to carry one very far. The reviewer's ignorance of Hebrew prevents him from offering a more detailed criticism.

A. NARASINGA RAO.

LE MYRE, Georges, *Le Baccara*—Hermann and Co., Editeurs, Paris. 1935, PL 204 Price 12 fr.

The famous gambling card game "Baccara" is dealt with, in the book under review, from the point of view of the Theory of Probability. The treatment of the subject is a very elementary one. Complete enumerations of the possible cases are made, and finally some "practical" rules stated. The book seems to have been written more for the frequenter of "casinos" than for the Mathematics student.

C. RACINE.

SOLUTIONS TO QUESTIONS

Authors sending Solutions to Questions (their own or of others) are requested to write them out separately in full sheets of paper in the form in which they usually appear in the STUDENT. Contractions should be avoided and absolutely necessary figures drawn neatly in Indian Ink on separate sheets. Letters and Numbers in diagrams should be in pencil.

Question 1674

(T. R. RAGHAVA SASTRY):—O is a fixed point in the plane of a $\triangle$ ABC. The reciprocal of ABC w.r.t. O is a $\triangle$ PQR. It is stated in Qn. 1631 that the centre of perspective of the $\triangle^s$ ABC, PQR for different radii of reciprocation trace out the rectangular hyperbola through ABCO.

Shew further that the axis of perspective of the two triangles envelops a parabola inscribed in ABC and having the line joining O to the orthocentre of ABC as directrix.

Solutions by C. T. Rajagopal and T. Raghavachari

(1) To find the reciprocal of a $\triangle$ ABC with respect to a circle centre O and radius k, we draw OX, OY, OZ perpendicular to BC, CA, AB respectively and on them take points A′, B′, C′ such that $OA' = k^2 / OX$ etc. Then $\triangle$ A′ B′ C′ is reciprocal to $\triangle$ ABC with respect to the circle.

It is obvious that, for various values of k, the position of A′ on OX and the corresponding positions of B′ on OY form two homographic ranges. Hence AA′, BB′ are corresponding rays of two homographic pencils. Therefore the locus of their intersections V, is a conic.

If $k=0$, V coincides with O. If $k=\infty$, V coincides with the orthocentre of △ ABC.

If B′ is on BA (and C′ on CA), V coincides with A etc. Thus the locus sought is a rectangular hyperbola through ABCO.

Reciprocating this result with respect to O we see that the envelope of the axes of perspective of △s ABC and its reciprocal PQR is a parabola whose direction passes thro' O. We have only to note that the fixed △ ABC now mentioned is different from the fixed △ ABC considered before.

(2) To get the reciprocal of the △ ABC with respect to O, take points L, M, N on OA, OB, OC such that $OL \cdot OA = OM \cdot OB = ON \cdot OC = k^2$ where k is the radius of reciprocation. Through L, M, N draw B′ C′, C′A′, A′B′ ⊥r to OA, OB, OC. A′B′C′ is the reciprocal triangle.

Now let B′C′ cut BC in P, and C′A′ cut CA in Q. It is clear that the straight lines (B′C′) and st. lines (C′A′) are rays of two homographic pencils with vertices at infinity. Therefore the ranges (P) on BC and (Q) on CA are homographic ranges. Similarly considering the range (P) on BC and the range (R) on AB, we see that the st. line PQR which is the axis of perspective, envelops a conic inscribed in the △ ABC. Making $k \to \infty$, we see that two corresponding positions of P, Q are at infinity on the respective st. lines BC, CA. Therefore the envelope is a parabola.

When $k=0$, line PQR is the ortholine of O with respect to ABC. Reciprocating w.r.t. a circle at O, the ortholine becomes the orthocentre of the reciprocal △ *abc* and the reciprocal of the parabola passes thro' this orthocentre, i.e., the reciprocal is a R.H. ∴ O lies on the directrix of the parabola.

Question 1730

(V. THÉBAULT)—With figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 taken each once, form a perfect square whose root is of the form $a\,a\,b\,c\,c$.

Solution by the proposer.

Since

$$0+1+2+3+4+5+6+7+8+9=45=M\ (9)$$

we should have

$$s = a + a + b + c + c = 2(a + c) + b = M\ (3)$$

which leads to

$$s = 9,\ 12,\ 15,\ 18,\ 21,\ 24,\ 27,\ 30,\ 33,\ 36,\ 39,\ \text{or}\ 42$$

Moreover, we have $31623 < N < 99999$

We have two solutions corresponding to $s = 15$ and $s = 30$

$$N^2 = 33144^2 = 1098524736$$
$$N^2 = 99066^2 = 9814072356$$

The second solution given above has also been obtained by trial by R. Sharangi.

1660. (V. RAMASWAMI AIYAR).—Two plane triangles ABC, A′B′C′ are in perspective. If A′B′C′ be moved rigidly and without rotation so as to be constantly in perspective with ABC,

(1) The locus of the centre of perspective as in the body of ABC is a conic X; and, as in the body of A′B′C′, is a conic X′;

(2) The locus of the Centre of the (instantaneous) Conic of reciprocation F of the two triangles, as in the body of ABC, is the conic X; and as in the body of A′B′C′ is the Conic X′;

(3) The Conics X, X′ have their points at infinity identical.

Solution by K. Satyanarayana

The result (5·1) of the paper on 'Some results connected with triangles in perspective, *Math. Student* Vol. II 49-59 shows that the locus of the centre of perspective is the conic X (denoted by $\sum$ in the paper). Relative to the $\triangle$ A′B′C′ $\triangle$ ABC receives displacement and so the locus as in the body of the $\triangle$ A′B′C′ must be X′ (denoted by $\sum'$ in the paper). The fact established in the paper that X, X′ ($\sum$, $\sum'$) are similar and similarly placed proves (3).

The centre of the conic for which the $\triangle^s$ are reciprocal is the point of intersection of $\sum$; $\sum'$ other than O. [Result (4·4)]. Hence (2) follows.

Question 1732

(A. MOËSSNER).—What is the general solution of the equations

$$A^n + B^n + C^n = D^n + E^n + F^n\ (n = 2,\ 4)$$

when we have also $\quad A^2 + B^2 = E^2; \quad D^2 + F^2 = C^2.$

Example $\quad 20^2 + 21^2 + 37^2 = 12^2 + 29^2 + 35^2$

$$20^4 + 21^4 + 37^4 = 12^4 + 29^4 + 35^4$$

while we have also

$$20^2 + 21^2 = 29^2$$
$$12^2 + 35^2 = 37^2.$$

Solution by T. Mahrenholz

Let $A = p^2 - q^2$, $B = 2pq$, $E = p^2 + q^2$, $D = r^2 - s^2$, $F = 2rs$, $C = r^2 + s^2$, then

$$\begin{aligned}(p^2 - q^2)^4 + 16p^4q^4 + (r^2 + s^2)^4 &= (p^2 + q^2)^4 + (r^2 - s^2)^4 + 16r^4 s^4,\\ 16p^4q^4 + 8r^6 s^2 + 8r^2 s^6 &= 8p^2q^6 + 8p^6q^2 + 16r^4 s^4,\\ r^6 s^2 - 2r^4 s^4 + r^2 s^6 &= p^2q^6 - 2p^4q^4 + p^6q^2,\\ r^2s^2 (r^2 - s^2)^2 &= p^2q^2 (p^2 - q^2)^2,\\ rs (r^2 - s^2) &= pq (p^2 - q^2).\end{aligned}$$

Now if $q = ra$, $s = pb$, $a = bc$, $c = 1 + z$ then

$$\frac{p^2}{r^2} = \frac{b^2c^3 + 1}{b^2 + c} = \frac{b^2 (1 + z)^3 + 1}{b^2 + z + 1} = (1 + dz)^2, \text{ provided}$$

$$z^3 (b^2 - d^2) + z^2\{3b^2 - 2d - (b^2 + 1) d^2\} + z\{3b^2 - 2d (b^2 + 1) - 1\} = 0.$$

For $d = b$ we have

$$z = \frac{2b^3 - 3b^2 + 2b + 1}{- b^4 + 2b^2 - 2b}, \qquad c = \frac{- b^4 + 2b^3 - b^2 + 1}{- b^4 + 2b^2 - 2b},$$

$$1 + dz = \frac{b^3 - 3b^2 + 4b - 1}{- b^3 + 2b - 2},$$

that is

$$\begin{aligned}p &= b^3 - 3b^2 + 4b - 1\\ q &= b^4 - 2b^3 + b^2 - 1\\ r &= b^3 - 2b + 2\\ s &= b^4 - 3b^3 + 4b^2 - b\end{aligned}$$

or if $b = x/y$

$$\begin{aligned}p &= y (x^3 - 3x^2y + 4xy^2 - y^3)\\ q &= x^4 - 2x^3y + x^2y^2 - y^4\\ r &= y (x^3 - 2xy^2 + 2y^3)\\ s &= x (x^3 - 3x^2y + 4xy^2 - y^3).\end{aligned}$$

x	y	p	q	r	s	A	B	C	D	E	F
1	2	2	− 5	6	1	− 21	− 20	37	35	29	12
3	1	11	35	23	33	− 1104	770	1618	− 560	1346	1518
1	3	3	− 77	111	1	− 5920	− 462	12322	12320	5938	222
2	3	51	− 77	78	34	− 3328	− 7854	7240	4928	8530	5304

An another way is

$$d=\frac{3b^2-1}{2(b^2+1)},\qquad z=-\frac{3(b^2+1)^3}{4b^6-b^4+10b^2-1},$$

$$c=\frac{b^6-10b^4+b^2-4}{4b^6-b^4+10b^2-1},\qquad 1+dz=\frac{b^6+17b^4-17b^2-1}{2(4b^6-b^4+10b^2-1)},$$

$$p=b^6+17b^4-17b^2-1$$
$$q=2b(b^6-10b^4+b^2-4)$$
$$r=2(4b^6-b^4+10b^2-1)$$
$$s=b(b^6+17b^4-17b^2-1)$$

Or taking $b=x/y$

$$p=y(x^6+17x^4y^2-17x^2y^4-y^6)$$
$$q=2x(x^6-10x^4y^2+x^2y^4-4y^6)$$
$$r=2y(4x^6-x^4y^2+10x^2y^4-y^6)$$
$$s=x(x^6+17x^4y^2-17x^2y^4-y^6).$$

For $x=2,\ y=1$ we have

$p=89,\ q=-128,\ r=186,\ s=178,$
A = 8463, B = 22784, C = 66280, D = 2912,
E = 24305, F = 66216.

Question 1737

(V. RAMASWAMI AIYAR)—If a point P and the feet of the perpendiculars from it on the sides of a quadrilateral be on a parabola the locus of P is a Cardioid whose cusp is at the Miquel point of the quadrilateral.

If we have five lines there will be five such cardioids. They have a pair of common points lying on the Miquel circle of the line.

Question 1738

(V. RAMASWAMI AIYAR)—Given a quadrilateral let P be any point and let Γ_p denote the conic passing through P and the feet of the perpendiculars from it on the sides of the quadrilateral.

(i) If Γ_p be a rectangular hyperbola, the locus of P is the centre circle of the quadrilateral.

(ii) The centre circle cuts the circumcircle of the triangle formed by any *three of the lines* in a point whose pedal line with reference to that triangle is parallel to the fourth line.

(iii) If Γ_p be a hyperbola whose asymptotes cut at a given angle the locus of P is a hyperbolic limacon whose node is at the Miquel point of the quadrilateral.

(iv) If Γ_p be an ellipse of given eccentricity the locus of P is an elliptic limacon having a conjugate point (imaginary node) at the Miquel point of the quadrilateral.

(iv*a*) For various given eccentricities or asymptotic angles the limacons form the system $r=a\cos\theta\pm b$ in polar co-ordinates where b varies. The origin is the Miquel point of the system and the circle $r=a\cos\theta$ is the centre circle of the quadrilateral.

(v) Given five lines the centre circles of these taken four at a time are concurrent at a point Z which lies on the Miquel circle of the five lines. The point of concurrence is the focus of the unique rectangular bi-parabola touching the five lines and is also the inverse of the centre of the Miquel circle in the centre circle of the five lines.

(vi) Given five lines five limacons are determined (under iv *a*) for a given eccentricity or asymptotic angle taking the lines four at a time. The limacons will be similar and they will have the common intersection lying on the Miquel circle.

(vi*a*) The limacons include the five cardioids as well.

(vii) The two common points lying on the Miquel circle (result vi) are equidistant from Z. Their joins form a system of parallel chords bisected by the diameter through Z (in theorem v.)

(viii) For points of the Miquel circle, the eccentricity of the pedal conic is *stationary* for the point diametrically opposite to Z in (v).

(ix) If $M_1, \ldots\ldots M_5$ be the Miquel points of the lines taken four at a time and $M_1T_1, \ldots M_5T_5$ be the chords of the Miquel circles touching the respective centre circles, at $M_1, \ldots\ldots M_5$, then the lengths of the chords are proportional to $M_1Z, \ldots\ldots M_5Z$.

Vide *J.I.M.S.* Vol. XVI, p. 282.

Solution by Faqir Chand Auluck.

Let $L_i \equiv l\sec\alpha_i - x\cos\alpha_i - y\sin\alpha_i$ $(i=1, 2, 3, 4)$ be four straight lines whose Miquel point is the origin. Let $2l$ be the semi-latus rectum of the parabola touching the four lines.

The Conic Γ_p given by

$$\begin{vmatrix} (x-x_0)^2 & (x-x_0)(y-y_0) & (y-y_0)^2 & x-x_0 & y-y_0 \\ p_1\cos^2\alpha_1 & p_1\cos\alpha_1\sin\alpha_1 & p_1\sin^2\alpha_1 & \cos\alpha_1 & \sin\alpha_1 \\ p_2\cos^2\alpha_2 & p_2\cos\alpha_2\sin\alpha_2 & p_2\sin^2\alpha_2 & \cos\alpha_2 & \sin\alpha_2 \\ p_3\cos^2\alpha_3 & p_3\cos\alpha_3\sin\alpha_3 & p_3\sin^2\alpha_3 & \cos\alpha_3 & \sin\alpha_3 \\ p_4\cos^2\alpha_4 & p_4\cos\alpha_4\sin\alpha_4 & p_4\sin^2\alpha_4 & \cos\alpha_4 & \sin\alpha_4 \end{vmatrix} = 0$$

where $p_i \equiv l\sec\alpha_i - x_0\cos\alpha_i - y_0\sin\alpha_i$ passes through the point (x_0, y_0) and the feet of the perpendiculars from (x_0, y_0) on the lines L_i, the foot of the perpendicular on L_i being $(x_0+p_i\cos\alpha_i,\ y_0+p_i\sin\alpha_i)$.

The coefficients $a, b, 2h$ of x^2, y^2 and xy are

$$\alpha(x_0^2+y_0^2)-\gamma l x_0-\beta l y_0,\ \alpha(x_0^2+y_0^2) \text{ and } -l\beta x_0+l\gamma y_0$$

respectively where α, β, γ are the determinants formed by omitting the first, second or third column respectively in the matrix

$$\left\|\begin{matrix} \sec\alpha_1\sin^2\alpha_1 & \sin^3\alpha_1 & \cos^3\alpha_1 & \cos\alpha_1 & \sin\alpha_1 \\ \cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot \\ \sec\alpha_4\sin^2\alpha_4 & \sin^3\alpha_4 & \cos^3\alpha_4 & \cos\alpha_4 & \sin\alpha_4 \end{matrix}\right\|$$

Γ_p is a rectangular hyperbola if (x_0, y_0) lies on the circle $x^2+y^2 = l(\gamma x+\beta y)/2\alpha$ which becomes, on rotating the axes through an angle $\tan^{-1}\beta/\gamma$

$$x^2+y^2=\frac{l\sqrt{(\beta^2+\gamma^2)}}{2\alpha}x=ax \text{ (say) so that } r=a\cos\theta. \quad \dots \quad (1)$$

If O_1 is the circumcentre of one of the four triangles, the conic through O_1 and the feet of the perpendiculars on the four lines is easily seen to be a rectangular hyperbola and therefore it lies on (1). Therefore (1) is the centre circle of the quadrilateral.

If the centre circle cuts the circumcircle of the triangle formed by three lines L_1, L_2, L_3 in Q, Γ_q degenerates into two perpendicular lines, the pedal line of Q with respect to the above triangle and a line perpendicular to L_1. Therefore the pedal line is parallel to L_4.

If
$$\frac{4(h^2-ab)}{(a+b)^2}=k,$$

the locus of (x_0, y_0) becomes, after rotating the axes through an angle $\tan^{-1}\beta/\gamma$

$$r=a\cos\theta\pm b \qquad \text{where } b=a/\sqrt{(1+k)}. \quad \dots \quad (2)$$

If Γ_p is a hyperbola, k is positive and therefore (2) is a hyperbolic limacon whose node is at the Miquel point of the quadrilateral. If Γ_p is an ellipse k is negative and therefore (2) is an elliptic limacon having a conjugate point at the Miquel point of the quadrilateral.

If Γ_p is a parabola, $k=0$, we get a cardioid whose cusp is at the Miquel point.

I have proved elsewhere* that if the five feet of the perpendiculars from a point on five given lines L_i $l=1, 2, 3, 4, 5$ lie on a conic passing through the point, the point lies on the Miquel circle C of the five lines. ... (3)

The Miquel circle C intersects one of the centre-circles C_1 in M_1 (the Miquel point corresponding to C_1) and in another point Z. By the

* *Indian Academy of Science,* Vol. V, No. 6.

above theorem (3) it follows that Z lies on all the 5 centre-circles of the five lines.

The limacon $r = a\cos\theta + b$ is the first positive pedal of

$$(x-a)^2 + y^2 = b^2 \qquad \ldots \quad (4)$$

with respect to the origin. Let M'_1 be the point diametrically opposite to M_1 on the Miquel circle. If the pair of tangents from M'_1 to the circle (4) intersect C in B_1 and B_2, it easily follows by the application of the theorem (3) that B_1 and B_2 lie on all limacons corresponding to the given value of k.

As the equations of the five limacons can be put in the form $r = a(\cos\theta + 1/\sqrt{(1+k)}$ where a varies. Hence the limacons are similar.

When $k=0$, we get the case of the cardioids.

It can be proved easily that M_1', Z, and the centre of the circle (4) are collinear and hence it follows that $B_1Z = B_2Z$. Chords B_1B_2, for various values of k are bisected by the diameter YZ through Z, where Y, is diametrically opposite to Z.

Now the pedal conic of a point P on the Miquel circle passes through P. If e is its eccentricity, then $b = \left(\frac{2}{e^2} - 1\right) a$. Let P be one of the common points.

If $P\hat{Y}Z = \theta$, then $e^2 = 2/(1 + \lambda \sin \theta)$ where $\lambda = M_1'G_1/a'$ where the point G_1, is the centre of the circle (4). It follows that e is stationary if $\theta = \pi/2$ i.e. at the point Y which is diametrically opposite to A.

As e is the same for the five limacons which pass through p, λ is also the same for all these.

i.e. M'_1G_1/a is the same for the five Miquel points,

or $\sin M'_1M_1G / \sin M_1M'_1G_1$ which is equal to M_1Z/M_1T_1 is constant Hence the result (ix) follows.

There is only one point Z whose pedal conic is a rectangular hyperbola. The first pedal of the unique rectangular bi-parabola touching the five lines, with respect to its focus is, again, a rectangular hyperbola.* Therefore Z coincides with the focus.

The locus of the centre of the circle C, which passes through Z and a point M on the Miquel circle and satisfying the condition

* Vide *J.I.M.S.* Vol. XVII, P. 77.

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JUNE, 1937

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